Czech Economic Review 8 (2014) 55–67
Acta Universitatis Carolinae Oeconomica

Aggregating and Updating Information

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Received 9 June 2014; Accepted 23 September 2014

Abstract We study information aggregation problems where to a set of measures a single measure of the same dimension is assigned. The collection of measures could represent the beliefs of agents about the state of the world, and the aggregate would then represent the beliefs of the population. Individual measures could also represent the connectedness of agents in a social network, and the aggregate would reflect the importance of each individual. We characterize the aggregation rule that resembles the Nash welfare function. In the special case of probability aggregation problems, this rule is the only one that satisfies Bayesian updating and some well-known axioms discussed in the literature.

Keywords Belief aggregation, belief updating, Nash welfare function
JEL classification C71, D63, D74

1. Introduction

We study information aggregation problems where to a set of measures a single measure of the same dimension is assigned. The collection of measures could represent the beliefs of agents about the state of the world, and the aggregate would then represent the beliefs of the population. Individual measures could also represent the connectedness of agents in a social network, and the aggregate would reflect the importance of each individual. We characterize axiomatically the aggregation rule (the *Nash rule*) that resembles the Nash welfare function (Kaneko 1979). In the context of probability aggregation the rule is sometimes called a logarithmic opinion pool (see Clemen and Winkler 1999).

This is done both in the case the measures are probability distributions and in the case of non-normalized measures. In the special case of probability aggregation problems, the Nash rule rule is the only one that satisfies Bayesian updating on top of some standard axioms. Genest (1984) has obtained this result but he assumes that the state space is infinite whereas we use finite state spaces. He also assumes a rather restrictive functional form, and because of that he essentially needs only one axiom in his characterization. Barrett and Pattanaik (1987) study the assumptions behind this functional form and are able to characterize this rule when the state space is finite (we discuss these papers at the end of Section 4). Finally, while probability measures seem natural in belief aggregation problems, non-normalized measures could be better suited in some network applications.

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Crès et. al (2011) and Gilboa et. al (2004) are recent papers where belief aggregation or belief *and* preference aggregation problems are studied. In Crès et. al (2011) there is a decision maker and a number of experts who all have the same utility function but different set of prior beliefs (probability measures) over the states. The problem is how to determine the beliefs for the decision maker in a reasonable way. Gilboa et. al (2004) study utilitarian aggregation of preferences and beliefs in the social choice context: when are society's welfare function and beliefs representable as weighted averages of those of individual agents.

The same machinery that for decades has been used to analyze social choice problems can be applied to all kinds of belief or opinion aggregation problems. Recently these tools have been applied to the analysis and construction of citation indices and internet search engines (see Palacios-Huerta and Volij 2004; Slutski and Volij 2006). For recent papers dealing with judgement aggregation from the logical point of view, see List and Polak (2010) or Nehring and Puppe (2010).

The paper is organized in the following way. In Section 2 the notation and aggregation rules are introduced. The axioms are introduced in Section 3. The main results are given in Section 4. Section 5 contains conclusions.

2. Preliminaries

Let S be a nonempty finite set. A *measure* μ on S satisfies (i) $\mu(E) \geq 0$, for each $E \subset S$; (ii) $\mu(\emptyset) = 0$; and (iii) $\mu(E \cup E') = \mu(E) + \mu(E')$ for all disjoint subsets $E, E' \subset S$. In this paper we assume that all subsets are measurable. We may denote the measure of singletons $\{s\}$ by $\mu(s)$ instead of $\mu(\{s\})$. So for example $\mu(E) = \sum_{s \in E} \mu(s)$. Given a measure μ on S and $E \subset S$, the *restriction* of μ to E is a measure $\mu_{|E}$ on S defined by $\mu_{|E}(A) = \mu(A \cap E)$ for every $A \subset S$.

Given a finite set N with $n > 0$ elements, and measures m_i, $i \in N$, an ordered n-tuple $m = (m_i)_{i \in N}$ is called a *profile of measures*. Given a profile m of measures, the inequality $m(E) < m(E')$ means $m_i(E) < m_i(E')$ for all $i \in N$; inequality $m_i < m'_i$ means $m_i(E) < m'_i(E)$ for all nonempty $E \subset S$; inequality $m < m'$ means $m_i < m'_i$ for all $i \in N$.

Let $supp(\mu) = \{s \in S \,|\, \mu(s) > 0\}$ be the *support* of a measure μ on S. A measure with an empty support is called a *null measure* and we denote it by μ^0. Given a profile $m = (m_i)_{i \in N}$ of measures on S, let $supp(m) = \{s \in S \,|\, m_i(s) > 0, \forall i \in N\}$ be the intersection of the supports of the measures m_i. If there is any risk of confusion we will state explicitly whether $supp(m)$ means the support of a single measure or a profile of measures.

An *aggregation problem* is a triple $P = (N, S, m)$, where N is a nonempty finite subset of natural numbers $\mathbb{N} = \{0, 1, \ldots\}$, S is a nonempty finite set, and $m = (m_i)_{i \in N}$ is a profile of measures m_i on S. We denote the set of all aggregation problems (or simply problems) by $\mathcal{P}$. We may study the subclass of problems with a common support ($supp(m_i) = supp(m_j)$, for all $i, j \in N$) denoted by $\mathcal{P}^{cs}$, and a special case of this, the problems with full support ($supp(m_i) = S$, for all $i \in N$) denoted by $\mathcal{P}^+$. If we want to study subclasses of problems with a given set of agents N or states S, we may denote these classes by $\mathcal{P}^N$, $\mathcal{P}^{N,S}$, $\mathcal{P}^{+,N}$ *e.t.c.*

An interpretation of the model is that N is the set of agents, S is the set of states of the nature, and m_i is the measure for agent i representing his beliefs about what is the true state s. Another possible interpretation is that S is a set of goods, and that each m_i is an additively separable utility function over S. Then $m_i(E)$ would be the utility from a bundle $E \subset S$ of goods.

A third interpretation is that N is the set of authors, S is the set of articles in academic journals, and $m_i(s)$ denotes the number of times author i has cited article s. More generally, since tastes and beliefs are opinions and citations reflect opinions as well, we may also say that the measures m_i represent the opinions of the agents.

An *aggregation rule* is a function f sucht that $f(P)$ is a measure on S for each aggregation problem $P = (N,S,m) \in \mathcal{P}$. Depending on the interpretation of the aggregation problem P, $f(P)$ may be interpreted as an aggregate belief of the society, or as a social preference, or as a "general opinion".

We say that a problem $P = (N,S,m)$ is a *probability aggregation problem*, if each m_i is a probability measure and $f(P)$ should also be a probability measure. Note that this subclass of problems is different than the ones defined above, since the definition also restricts the class of feasible rules.

2.1 Some well-known aggregation rules

The *Average rule* f^A is the best known rule. It is defined by $f^A(P)(s) = \frac{1}{n}\sum_{i\in N} m_i(s)$ for every $s \in S$, for each problem $P = (N,S,m)$. This is sometimes called the *linear opinion pool* (see Clemen and Winkler 1999).

The Median rule f^M is defined as follows for every problem $P = (N,S,m)$ (see e.g. Balinski and Laraki 2007; Barthelemy and Monjardet 1981). Given $s \in S$, let $f^M(P)(s)$ be the median of the components of the vector $m(s)$. In case where the successive elimination of greatest and least values of the coordinates of the vector $m(s)$ leaves us with two components $m_i(s)$ and $m_j(s)$, we define the median to be the average of these values. For example, if $m(s) = (1,1,3)$, then the median is 1, but if $m(s) = (1,1,3,3)$, then the median is 2.

The *Borda rule* f^B is also quite well-known (see e.g. Nurmi and Salonen 2008; Saari 2006; Young 1974). Let $b_i(P)(s) = |\{s' \in S \,|\, m_i(s') \leq m_i(s)\}|$ for all $s \in S$, and let $f^B(P)(s) = \frac{1}{n}\sum_{i\in N} b_i(P)(s)$, for all problems $P = (N,S,m)$ (here $|X|$ means the cardinality of the set X). Note that if for each i the measures $m_i(s)$ are different for different states s, we get the standard form of the Borda rule. The Borda rule is often defined as the sum $\sum_{i\in N} b_i(P)(s)$. For all practical purposes the two versions are the same.

The *Nash rule* f^G is based on the Nash welfare function (Kaneko 1979), and the idea can be applied in the present context as well (this rule is sometimes called the *logarithmic opinion pool*, see Clemen and Winkler (1999). It is defined by $f^G(P)(s) = \sqrt[n]{\prod_{i\in N} m_i(s)}$ for each $s \in S$, for each problem $P = (N,S,m)$. The superscript G refers to the fact that $f^G(P)(s)$ is the geometric average of the individual $m_i(s)$-values.

The *Norm rules* f^{EN}, f^{SN} are based on the Euclidean norm and *sup*-norm, respectively. The rule f^{EN} is defined by $f^{EN}(P)(s) = n^{-1/2}\sqrt{m_1(s)^2 + \cdots + m_n(s)^2}$ for each $s \in S$, for each problem $P = (N,S,m)$. Define f^{SN} by $f^{SN}(P)(s) = \sup\{|m_1(s)|, \ldots,$

$|m_n(s)|\}$ for each $s \in S$, for each problem $P = (N,S,m)$ (here $|x|$ means the absolute value of number x). Note that the norm rule corresponding to the city block norm $|m_1(s)| + \cdots + |m_n(s)|$ is the Average rule f^A.

The rules defined above can be defined in such a way that they are applicable in probability aggregation problems as well. There are many ways to do it. Suppose the subclass of problems is such that $f(P)(S) > 0$, and each m_i is a probability measure, for every problem $P = (N,S,m)$ in this subclass. Then a probability aggregation rule $f^{\times}$ can be defined by $f^{\times}(P)(s) = f(P)(s)/f(P)(S)$ for every $s \in S$. We call $f^{\times}$ the *multiplicative normalization* of f.

3. Properties of aggregation rules

Now we present some properties or axioms that aggregation rules could satisfy. For a more comprehensive treatment of different aggregation procedures and their properties, see e.g. Nurmi (2002).

We do not specify in each case the subclass of problems where the axiom in question should be applicable. Instead, we specify in our theorems the subclass where the rules are defined, and axioms are then restricted to this subclass as well. This way we may use the axioms in a more flexible manner. For example, if we analyze the class of problems with full support, then *Regularity* (defined below) has no bite. Notable exception to this practice is the axiom *Bayesian updating* that is designed specifically for probability aggregation problems.

Given agent sets N and M with equally many members, let $\pi : N \longrightarrow M$ be any bijection, and given an n-tuple m of measures, let πm be an n-tuple of measures such that $\pi m_{\pi(i)} = m_i$. In other words, the agent $\pi(i)$ has the same measure in profile πm as person i has in profile m. Given an aggregation problem $P = (N,S,m)$ and a bijection $\pi : N \longrightarrow M$, define another aggregation problem $Q = (M,S,\pi m)$, which is otherwise the same as P except that agent $\pi(i) \in M$ has been given the measure m_i of agent $i \in N$: $\pi m_{\pi(i)} = m_i$.

Axiom 1 (*Anonymity*, AN)**.** For every bijection $\pi : N \longrightarrow M$ and aggregation problems $P = (N,S,m)$ and $Q = (M,S,\pi m)$, it holds that $f(Q) = f(P)$.

Let S and T be two finite sets with the same number of elements. Given an aggregation problem $P = (N,S,m)$ and a bijection $\pi : S \longrightarrow T$, define another aggregation problem $Q = (N,T,m^{\pi})$, which is otherwise the same as P except that elements $s \in S$ are replaced by elements $\pi(s) \in T$, and $m^{\pi}(\pi(s)) = m(s)$, for all $s \in S$.

Axiom 2 (*Neutrality*, NE)**.** For every bijection $\pi : S \longrightarrow T$ and aggregation problems $P = (N,S,m)$ and $Q = (N,T,m^{\pi})$, it holds that $f(P)(s) = f(Q)(\pi(s))$ for every $s \in S$.

Anonymity says that the labels of the agents do not matter, while *Neutrality* says that labels of the states do not matter. All the rules defined in Section 2.1 satisfy *Neutrality* and *Anonymity*. These rules satisfy also the following axiom called *Unanimity*.

Axiom 3 (*Unanimity*, UN)**.** If $m_1 = \cdots = m_n = \mu$ in an aggregation problem $P = (N,S,m)$, then $f(P) = \mu$.

The previous three axioms are standard in the literature.

Axiom 4 (*Common scale covariance*, CSC)**.** If $P = (N,S,p)$ and $Q = (N,S,q)$ are two problems such that $q = ap$ for some $a > 0$, then $f(Q) = af(P)$.

Common scale covariance says that if we multiply the opinions of all agents by the same constant then the aggregate opinion will be multiplied by the same constant. This property is sometimes called *Homogeneity* (of degree one). All rules in Section 2.1 except the Borda rule satisfy this axiom.

Axiom 5 (*Individual scale covariance*, ISC)**.** Suppose problems $P = (N,S,p)$ and $Q = (N,S,q)$ are such that for some $i \in N$, $q_i = ap_i$ for some $a > 0$, and $q_j = p_j$ for all $j \neq i, j \in N$. Then $f(Q) = \alpha_i^P(a)f(P)$ for some strictly increasing and continuous function $\alpha_i^P : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$.

Individual scale covariance says that if we multiply agent i's beliefs by a positive constant, then the aggregated beliefs are also multiplied by some positive constant. This axiom is needed in applications where only the ratios $m_i(s)/m_i(s')$ and $f(P)(s)/f(P)(s')$ of individual and aggregate opinions matter. Because if ISC is satisfied, scaling the measure m_i up or down has no effect on these ratios. CSC does not guarantee this unless all measures m_i are multiplied by the same constant. On the other hand, if all measures are multiplied by the same constant $a > 0$, then applying ISC iteratively we can see that the aggregated beliefs are multiplied by *some* constant b, but not necessarily by a.

All rules in Section 2.1 except the Borda rule satisfy CSC but only the Nash rule f^G satisfies both ISC and CSC. Axioms CSC and ISC are of course not applicable in probability aggregation problems.

The next axiom is a version of the well-known property the appears already in Arrow's seminal work (Arrow 1963).

Axiom 6 (*Independence of irrelevant alternatives*, IIA)**.** Let $P = (N,S,p)$ and $Q = (N,S,q)$ be two aggregation problems such that $p(s) = q(s)$ and $p(s') = q(s')$ for some $s,s' \in S$. Then $f(P)(s) < f(P)(s')$ if and only if $f(Q)(s) < f(Q)(s')$.

All the rules defined in Section 2.1 satisfy this axiom except the Borda rule. The following axiom is closely related to **IIA**.

Axiom 7 (*Updating*, UP)**.** If $P = (N,S,m)$ and $Q = (N,E,m_{|E})$ are such that $E \subset S$ and $E \neq \emptyset$, then $f(Q) = f(P)_{|E}$.

We will show in Section 4 that every rule that satisfies UP satisfies also **IIA**. The only rule in Section 2.1 that does not satisfy UP is the Borda rule. The following axiom is the well-known Bayesian updating property. It's domain is the class of probability aggregation problems.

Axiom 8 (*Bayesian updating*, BUP)**.** Let $P = (N,S,p)$ and $Q = (N,E,q)$ be two probability aggregation problems such that $E \subset S$, $p_i(E) > 0$ and q_i is derived from p_i by the Bayes rule for all $i \in N$. Then the probability measure $f(Q)$ is derived from the probability measure $f(P)$ by the Bayes rule.

Bayesian updating seems so natural that one may wonder whether it has any bite at all. However, even the best-known aggregation rule, the *Average rule*, fails to satisfy this axiom. This axiom is sometimes called *external Bayesianism* (see Genest 1984; Clemen and Winkler 1999). The following axiom makes sense in all kinds of aggregation problems.

Axiom 9 (*Expert proofness*, EP)**.** Let $P = (N, S, p)$ and $Q = (N \setminus \{i\}, S, q)$ be two aggregation problems such that $i \in N$, $p_j = q_j$, for all $j \in N \setminus \{i\}$, and $p_i = f(Q)$. Then $f(P) = f(Q)$.

If agent i adopts the *aggregated opinions* of the other agents $j \in N \setminus \{i\}$, then the aggregated opinions of the enlarged population N are the same as the aggregated opinions of $N \setminus \{i\}$. One interpretation is that the public already has a quite good idea of what the opinions in the society are, and they may have adjusted a little bit their own views as a response. Then if an expert comes and makes the society's opinions common knowledge, the public has no reason to adjust their opinions any more. We will show in Section 4 that all the rules defined in Section 2.1 satisfy EP.

Related properties such as *Consistency* and *Positive involvement* have been discussed in the context of social choice functions (see Young 1974, and Saari 1995). Suppose N and M are disjoint sets of agents who have beliefs over the same state space S. In our context Consistency could be formulated as follows. If the aggregated beliefs of both groups is that state $\bar{s}$ has the greatest measure among all states $s \in S$, then the group $N \cup M$ would also have this same aggregated belief about $\bar{s}$. Aggregation rule would satisfy Positive involvement, if this conclusion would hold in case all agents i in group M would have identical beliefs m_i, and $m_i(\bar{s}) \geq m_i(s)$ for all $s \in S$.

Expert proofness says that aggregated beliefs do not change if a new agent enters with beliefs that are equal to the aggregated beliefs of the rest of the society. Expert proofness is silent about what would happen if the new agent had the same beliefs as at the rest of the society about states with the highest measure only. Positive involvement would imply that the aggregated beliefs about these states would not change. Consistency would also imply this, provided that aggregated beliefs of single agent societies coincide with the beliefs of that agent (this property is called *Faithfulness* in Young (1974)).

The last axiom discussed here is a rather weak version of unanimity: if all agents agree that state s has zero measure, then the aggregated beliefs must also put zero measure on s.

Axiom 10 (*Regularity*, REG)**.** Given $P = (N, S, m)$, it holds that $f(P)(s) = 0$ for a given $s \in S$, if $m_i(s) = 0$ for all $i \in N$.

Regularity is satisfied in the class $\mathcal{P}$ by all the other rules defined in Section 2.1.

4. Results

All the rules defined in Section 2.1 are *Expert proof.*

Lemma 1. *The Average rule, the Borda rule, the Median rule, the Nash rule, and the Norm rules f^{EN} and f^{SN} satisfy EP.*

Proof. See the Appendix.

We show next that the axiom UP implies IIA.

Lemma 2. *If a rule f satisfies UP, then it satisfies IIA.*

Proof. Suppose f satisfies UP. Let $P=(N,S,p)$ and $Q=(N,S,q)$ be two aggregation problems as in the statement of IIA: $p(s)=q(s)$ and $p(s')=q(s')$ for two members $s,s'\in S$. Let $E=\{s,s'\}$, and $P'=(N,E,p_{|E})$, and $Q'=(N,E,q_{|E})$. Then by UP, $f(P')=f(P)_{|E}$ and $f(Q')=f(Q)_{|E}$. But $P'=Q'$ because $p_{|E}=q_{|E}$, and therefore f satisfies IIA. □

The next lemma is needed in the proofs of the main results.

Lemma 3. *Suppose f satisfies NE, ISC, and UP. Then the function α_i in the axiom ISC does not depend on P.*

Proof. Suppose problems $P=(N,S,p)$ and $Q=(N,S,q)$ are such that for some $i\in N$, $q_i=ap_i$ for some $a>0$, and $q_j=p_j$ for all $j\neq i, j\in N$. Given $s\in S$, define $P^s=(N,\{s\},p_{|\{s\}})$ and $Q^s=(N,\{s\},q_{|\{s\}})$. By UP, $f(P)(s)=f(P^s)(s)$ and $f(Q)(s)=f(Q^s)(s)=\alpha_i^P f(P^s)(s)$.

Let $P'=(N,X,p')$ and $Q'=(N,X,q')$ be such that (i) for some $x\in X$, $p'(x)=p(s)$, (ii) $q_i'=ap_i'$, and $q_j=p_j$ for all $j\neq i, j\in N$. Define $P'^x=(N,\{x\},p'_{|\{x\}})$ and $Q'^s=(N,\{x\},q'_{|\{x\}})$. Then by NE, $f(P'^x)=f(P^s)$ and $f(Q'^x)=f(Q^s)=\alpha_i^P f(P^s)$. By UP, $f(P')(x)=f(P'^x)(x)$, and $f(Q')(x)=f(Q'^x)(x)=\alpha_i^P f(P')(x)$. By ISC, $f(Q')=\alpha_i^{P'} f(P')$. But then $\alpha_i^P=\alpha_i^{P'}$, and we are done. □

We give next an axiomatic characterization of the Nash rule on the class of full support aggregation problems $\mathcal{P}^+$. First we characterize a one-parameter family of rules.

Theorem 1. *Let f be a rule satisfying AN, ISC, CSC, NE, and UP on the class of full support problems $\mathcal{P}^{+,N}$ with a given set N of agents. Then for some $a>0$, $f=af^G$, or $f(P)$ is the null measure μ^0 for all $P\in\mathcal{P}^{+,N}$.*

Proof. See the Appendix.

Remark 1. If $f(P)=\mu^0$ for all P, then $f=0\cdot f^G$, so the theorem gives a characterization of a one-parameter family $\mathcal{F}=\{af^G \mid a\geq 0\}$ of rules.

Remark 2. Theorem 1 does not say whether or not the parameter a of the family $\mathcal{F}$ depends on N.

Remark 3. The axiomatization is tight in the sense that if we drop any of the axioms, then there are many more rules that satisfy the remaining axioms. The only axiom for which this is not obvious is UP. The following rule satisfies all the other axioms except UP.

$$f(P) = \left[\prod_i m_i(S)\right]^{1/n} \left[\frac{1}{n}\sum_i \left(\frac{m_i}{m_i(S)}\right)\right]$$

If we add *Unanimity* to the list of axioms of Theorem 1, the only possible solution is the Nash rule f^G and the agent set N need not be the same in every problem. Moreover, we can replace the full support assumption by the common support assumption.

Lemma 4. *If a rule f satisfies UP and UN, then f satisfies REG.*

Proof. Let $P = (N,S,m)$ be such that $m_i(s) = 0$ for all $i \in N$, for some $s \in S$. Then by UP, $f(P')(s) = f(P)(s)$, where $P' = (N;\{s\},m_{|\{s\}})$. By UN, $f(P')(s) = 0$. □

Lemma 4 implies that in common support problems $P = (N,S,m)$, $f(P)(s) = 0$ for all $s \notin supp(m)$.

Theorem 2. *Let f be a rule satisfying AN, ISC, CSC, NE, UN and UP on the class of common support problems $\mathcal{P}^{cs}$, then $f = f^G$.*

Proof. If $m_1 = \cdots = m_n = \mu$ in equation (1), then by UN we get that $f(Q^N) = 1$. Since this holds independently of N, we are done. □

Remark 4. The Nash rule f^G satisfies all the axioms mentioned in Theorem 2 in the class $\mathcal{P}$ of all problems. At the moment I do not know if there are other rules satisfying these axioms in the class $\mathcal{P}$.

Here is our main result concerning probability aggregation problems. Let $f^{G\times}$ be the multiplicative normalization of the Nash rule f^G.

Theorem 3. *Suppose f is a rule that satisfies AN, ISC, CSC, NE, UN and UP on the class of common support problems $\mathcal{P}^{cs}$, and that its multiplicative normalization $f^\times$ satisfies BUP on the class of probability aggregation problems in $\mathcal{P}^{cs}$. Then $f = f^G$ and $f^\times = f^{G\times}$.*

Proof. See the Appendix.

Genest (1984) assumes infinite state space S and that there is a function $F : [0,\infty)^n \longrightarrow [0,\infty)$ such that $f(P)(E) = F(m_1(E),\ldots,F_n(E))$ for each measurable $E \subset S$. This kind of formalization hides behind it many assumptions. Barrett and Pattanaik (1987) formalize some of these assumptions as axioms, and drop the assumption that S is infinite. They assume monotonicity (also for conditional probabilities) which says that $p_i'(s) \geq p_i(s)$ for all i implies $f(P')(s) \geq f(P)(s)$, where $P = (N,S,p), P' = (N,S,p')$.

5. Conclusions

The main contributions of this paper are (i) to propose new axiomatic characterizations of the Nash rule, and (ii) to give a detailed analysis of the relationships between different axioms and aggregation rules.

The axioms *Anonymity*, *Neutrality*, *Common scale covariance*, and *Unanimity* are quite uncontroversial in the context of belief aggregation. In the class of common support problems only the Nash rule satisfies these axioms together with *Individual scale covariance* and *Updating* (Theorem 2).

Individual scale covariance says that if the beliefs of an agent are multiplied by a positive constant, then the aggregated beliefs will also be multiplied by some positive constant. In other words, as long as the ratios $m_i(s)/m_i(s')$ do not change, the ratios $f(P)(s)/f(P)(s')$ of aggregated beliefs do not change either. This property seems most natural if the relevant information about beliefs is contained in the ratios $m_i(s)/m_i(s')$.

Updating requires that if the state space shrinks from S to $E \subset S$, and agents' updated beliefs are their original beliefs restricted to E, then the aggregated beliefs on E are the original aggregated beliefs restricted to E.

By Theorem 3 the normalized Nash rule is the only rule that satisfies the axiom *Bayesian updating* in addition to the axioms mentioned above. This axioms says that if agents apply Bayesian updating when they get more accurate information, then Bayesian updating is applied to the aggregated beliefs as well.

Bayesian updating is a very natural assumption in single person decision problems. It is quite surprising how much bite it has when applied to aggregation problems. This holds of course for many other versions and modifications of the "Independence of irrelevant alternatives" axiom.

Acknowledgment The author is grateful to Hannu Nurmi and seminar participants at the AMES 2013 meeting in Singapore for discussions and useful comments. The comments of three anonymous referees are gratefully acknowledged. I thank the Academy of Finland, the Yrjö Jahnsson Foundation, and the OP-Pohjola Group Research Foundation for financial support.

References

Arrow, K. J. (1963). *Social Choice and Individual Values*. New York, Wiley.

Balinski, M. and Laraki, R. (2007). A Theory of Measuring, Electing, and Ranking. *Proceedings of the National Academy of Sciences of the United States of America*, 104, 8720–8725.

Barrett, C. R. and Pattanaik, P. K. (1987). Aggregation of Probability Judgements. *Econometrica*, 55, 1237–1241.

Barthelemy, J. P. and Monjardet, B. (1981). The Median Procedure in Cluster Analysis and Social Choice Theory. *Mathematical Social Sciences*, 1, 235–267.

Clemen, R. T. and Winkler, R. L. (1999). Combining Probability Distributions From Experts in Risk Analysis. *Risk Analysis*, 19, 187–203.

Crès, H., Gilboa, I. and Vieille, N. (2011). Aggregation of Multiple Prior Opinion. *Journal of Economic Theory*, 146, 2563–2582.

Genest, C. (1984). A Characterization Theorem for Externally Bayesian Groups. *The Annals of Statistics*, 12, 1100–1105.

Gilboa, I., Samet, D. and Schmeidler, D. (2004). Utilitarian Aggregation of Beliefs and Tastes. *Journal of Political Economy*, 112, 932–938.

List, C. and Polak, B. (2010). Introduction to Judgement Aggregation. *Journal of Economic Theory*, 145, 441–466.

Kaneko, M. (1979). The Nash Social Welfare Function. *Econometrica*, 47, 423–435.

Nehring, K. and Puppe, C. (2010). Justifiable Group Choice. *Journal of Economic Theory*, 145, 583–602.

Nurmi, H. (2002). *Voting Procedures under Uncertainty*. Berlin-Heidelberg-New York, Springer.

Nurmi, H. and Salonen, H. (2008). More Borda Count Variations for Project Assesment. *Czech Economic Review*, 2, 109–122.

Palacios-Huerta, I. and Volij, O. (2004). The Measurement of Intellectual Influence. *Econometrica*, 72, 963–977.

Saari, D. (2006). Which Is Better The Condorcet or Borda Winner. *Social Choice and Welfare*, 26, 107–129.

Saari, D. (1995). *Basic Geometry of Voting*. Berlin-Heidelberg-New York, Springer.

Slutski, G. and Volij, O. (2006). Scoring of Web Pages and Tournaments. *Social Choice and Welfare*, 26, 75–792.

Young, H. P. (1974). An Axiomatization of Borda's Rule. *Journal of Economic Theory*, 9, 43–52.

Appendix

Proof of Lemma 1. Let $P = (N,S,p)$ and $Q = (N \setminus \{i\},S,q)$ be two problems as in the axiom EP. It is straightforward to verify that the Average rule satisfies EP.

Take the Borda rule f^B and consider the problems $P' = (N,S,b(P))$ and $Q' = (N,S,b(Q))$, where $b_i(P)$ is the measure derived from agent i's Borda scores $b_i(P)(s)$ in the problem $P = (N,S,p)$, and $b_j(Q)$ is derived from agent j's Borda scores $b_j(Q)(s)$ in the problem $Q = (N \setminus \{i\},S,q)$.

Since $p_i(s') < p_i(s)$ iff $b_i(P)(s') < b_i(P)(s)$, the problem P' is obtained from P by applying strictly increasing transformations to the measures p_i. Similarly, Q' is obtained from Q by applying strictly increasing transformations to the measures q_j.

Since the Borda scores b_i depend only on the ordinal ranking of states s, the individual Borda scores satisfy $b_i(P')(s) = b_i(P)(s)$ and $b_j(Q')(s) = b_j(Q)(s)$.

Since $p_i(s) = f^B(Q)(s) = \frac{1}{n-1}\sum_{j\in M} b_j(Q)(s)$ and $b_j(P) = b_j(Q)$ for every $j \in M$, we have

$$\frac{1}{n}\sum_{j\in N} b_j(P)(s) = \frac{1}{n}\Big[\sum_{j\in M} b_j(Q)(s) + \frac{1}{n-1}\sum_{j\in M} b_j(Q)(s)\Big],$$

which implies $f^B(P')(s) = f^B(Q')(s)$. Since $b_i(P')(s) = b_i(P)(s)$ and $b_j(Q')(s) = b_j(Q)(s)$, we have that $f^B(P)(s) = f^B(Q)(s)$, so the Borda rule satisfies EP.

The Median rule has the property that for all $s \in S$, $p_i(s) = f^M(Q)(s)$ and this is the median of the vector $q(s)$. But then $p_i(s)$ is the median of the coordinates of the vector $p(s)$ as well, and so f^M satisfies EP.

Let $P = (N,S,p)$ and $Q = (N\setminus\{i\},S,q)$ be as stated in EP, and define $M = N\setminus\{i\}$. Then for each $s \in S$,

$$f^G(P)(s) = \left[\prod_{j\in M} p_j(s)\left(\prod_{j\in M} p_j(s)\right)^{1/(n-1)}\right]^{1/n} = f^G(Q)(s),$$

and therefore the Nash rule satisfies EP.

If we use the Norm rule f^{EN}, we have

$$f^{EN}(Q)(s) = (n-1)^{-1/2}\sqrt{\sum_{j\neq i} m_j(s)^2},$$

which implies

$$f^{EN}(P)(s) = n^{-1/2}\sqrt{\sum_{j\neq i} m_j(s)^2 + (n-1)^{-1}\sum_{j\neq i} m_j(s)^2},$$

but then $f^{EN}(P)(s) = f^{EN}(Q)(s)$ as desired.

The proof for *sup*-norm rule f^{SN} is easy and omitted. □

Proof of Theorem 1. Clearly the rule that assigns the null measure to every problems satisfies these axioms. So suppose f is another rule satisfying the axioms AN, ISC, CSC, NE and UP.

Let $P = (N,S,m) \in \mathcal{P}^{+,N}$ be any problem and take any $s \in S$. By UP, $f(Q) = f(P)_{|\{s\}}$ where $Q = (N,\{s\},m_{|\{s\}})$. By the full support assumption, $m_i(s) > 0$ for every $i \in N$.

Let q^i be the vector such that $q^i_i = 1$ and $q^i_j = m_j(s)$ for all $j \neq i$, and let $Q^i = (N,\{s\},q^i)$. Note that $m_i(s) = m_i(s)q^i_i$. Then by ISC, $f(Q) = \alpha_i(m_i(s))f(Q^i)$, where α_i is the continuous strictly increasing function in the axiom ISC. By Lemma 3, α_i does not depend on P.

Let q^{ij} be the vector such that $q^{ij}_i = q^{ij}_j = 1$ and $q^{ij}_k = m_k(s)$ for all $k \neq i,j$, and let $Q^{ij} = (N,\{s\},q^{ij})$. Then $f(Q) = \alpha_j(m_j(s))\alpha_i(m_i(s))f(Q^{ij})$ by ISC.

Let P' be a problem that is otherwise like P except in problem P' player i has a measure $m'_i = m_j$ and player j has the measure $m'_j = m_i$. Derive Q' from P' in the same ways as Q was derived from P above, and construct q'^{ij} in the same fashion as q^{ij}.

By AN, $f(P) = f(P')$ and $f(Q) = f(Q')$, and therefore we must have $\alpha_j(m_j(s))\alpha_i(m_i(s)) = \alpha_j(m_i(s))\alpha_i(m_j(s))$. Since $m_i(s)$ and $m_j(s)$ are arbitrary positive numbers, and functions α_i and α_j are strictly increasing, we must have that $\alpha_i = \alpha_j$. Since players i and j were arbitrarily chosen, $\alpha_1 = \cdots = \alpha_n \equiv \alpha$.

Applying ISC recursively, we get that $f(Q) = \prod_i \alpha(m_i(s)) f(Q^N)$, where $Q^N = (N, \{s\}, \mathbf{1})$ and $\mathbf{1} = (1, \ldots, 1)$. In the special case $m_1(s) = \cdots = m_n(s) = a$, we get that $f(Q) = \alpha(a)^n f(Q^N)$. But by CSC, we must have $\alpha(a)^n = a$, or equivalently $\alpha(a) = \sqrt[n]{a}$.

It follows that

$$f(P)(s) = \left[\sqrt[n]{\prod_{i=1}^{n} m_i(s)}\right] f(Q^N). \tag{1}$$

Now the value $f(Q^N)$ must be the same for all $s \in S$ by NE, so $f(Q^N)(s) = a$, for some $a > 0$, for all $s \in S$. But the constant a must be the same for all problems $P' = (N, S', m')$.

To see this, note that in the axiom ISC the functions α_i of agents $i \in N$ were defined to be the same for all problems. In particular, α_i did not depend on the profile of measures m or the state space S. If we have some other problem $P' = (N, S', m')$, then by UP and NE, we get again that equation (1) holds, when m_i is replaced by m'_i and $Q^N = (N, \{s\}, \mathbf{1})$ is replaced by $Q'^N = (N, \{s'\}, \mathbf{1})$. But NE implies that these can be viewed as the same problem and hence they must have the same solution, so $f(Q^N) = f(Q'^N)$. So the values $f(P)$ and $f(P')$ are different only if m and m' are different. □

Proof of Theorem 3. It follows from Theorem 2 that $f = f^G$, so we just have to show that its multiplicative normalization $f^{G\times}$ satisfies BUP.

Take any probability aggregation problem $P = (N, S, p)$ with a common support. By Lemma 4 and UP, we may assume $S = supp(p)$. For any $s \in S$ we have $p_i(s) > 0$, and so $f^G(P)(s) = \left[\prod_i p_i(s)\right]^{1/n} > 0$ and $f^G(P)(A) = \sum_{s \in A} \left[\prod_i p_i(s)\right]^{1/n} > 0$ for every $A \subset S$. By the definition of the multiplicative normalization we have for any $s \in S$

$$f^{G\times}(P)(s) = \frac{f^G(P)(s)}{f^G(P)(S)}.$$

Now update $f^{G\times}(P)$ on the nonempty subset $E \subset S$ by using the Bayes rule:

$$f^{G\times}(P)(s \mid E) = \frac{f^G(P)(s)/f^G(P)(S)}{f^G(P)(E)/f^G(P)(S)}) = \frac{f^G(P)(s)}{f^G(P)(E)} \tag{2}$$

Let $Q = (N, E, q)$ be related to P as in the axiom BUP. So q is derived from p by applying the Bayes rule: $q_i(s) = p_i(s)/p_i(E)$, for all $i \in N$, for all $s \in E$. Therefore

$f^G(P)$ is computed by

$$f^G(Q)(s) = \frac{\left[\prod_i p_i(s)\right]^{1/n}}{\left[\prod_i p_i(E)\right]^{1/n}}, \ \forall s \in E.$$

The corresponding multiplicative normalization is computed by

$$f^{G\times}(Q)(s) = \frac{\left[\prod_i p_i(s)\right]^{1/n} \Big/ \left[\prod_i p_i(E)\right]^{1/n}}{\sum_{s\in E} \left[\prod_i p_i(s)\right]^{1/n} \Big/ \left[\prod_i p_i(E)\right]^{1/n}}, \ \forall s \in E. \tag{3}$$

But the right hand sides of equations 2 and 3 are the same. Therefore $f^{G\times}$ satisfies BUP. □